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EFFECTS OF GEOMETRY AND UNIDIRECTIONAL BODY FORCES ON THE STABILITY OF LIQUID LAYERS

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by

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SUMMARY

The stability of liquid layers with a prescribed interface geometry in the presence of surface tension and unidirectional body forces is investigated theoretically by means of a small vibration analysis. The fluid of the layers is assumed to be incompressible and inviscid and the flow irrotational. Particular emphasis is given to the effects of geometry and unidirectional body forces on the stability of such layers. They are studied by means of a semi-inverse method which allows the exact determination of particular eigenvibrations.

Following the classical method of investigating the stability of systems, we establish the equations of motion for small perturbations of the equilibrium configuration and formulate the boundary conditions for both two- and three-dimensional layers with interface surfaces having a constant mean curvature. We linearize the pertinent equations and express them in terms of the velocity potential. By separating the time and space variables, the vibration problem is reduced to an eigenvalue problem. Exact eigenvibrations are sought by considering solutions to Laplace's equation which contain a number of free parameters. The arbitrariness of these parameters is reduced by enforcing the solution to satisfy the interface boundary condition. Integrating the differential equation of the streamlines defined by such solutions and requiring the solid supporting surface to constitute a streamline, we arrive at rigorous vibration solutions. If we introduce sufficiently many free parameters we are in a position to review exact vibration solutions corresponding to a family of liquid layers with the same interface surface and different solid supporting surfaces and thus different thickness distributions.

The stability criteria of the layers are presented in terms of inequalities involving the magnitude of either the unidirectional body force, g, or the nondimensional stability parameter $\mu = T/\rho g a^2$ in which T denotes the surface tension, ρ the fluid density and "a" a characteristic length of the liquid layer. It is shown that the variation of the thickness of a fluid layer may significantly influence its stability, especially for body forces in the near-zero region. The

results also reveal that for some solid supporting surfaces, layers of the type considered are only stable if the body forces are larger than a certain positive value, while for other surfaces stability prevails as long as the body force exceeds a certain negative limit value. This fact may be utilized in the design of containers of fluid for near-zero gravity conditions in such a manner that the fluid adheres to a certain area of the container.

The interpretation of the results obtained in this analysis is subject to the restriction that the dimensions of the interface surface be large compared with the effective range of the contact forces, since no contact angle considerations have been introduced in explicit form.

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NOTATION

A, B, C, D, E	Constant coefficients
A_n, B_n, D_n	Coefficients of infinite or finite series
$F(t)$	Arbitrary function of time
$F(r,z), X(x), Z(z)$	Functions introduced to separate variables
H	Mean curvature $H = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$
J_m	m^{th} order Bessel function of first kind
$P_{p_n}^m$	Associated Legendre functions of degree m and order p_n
R_1, R_2	Principal radii of curvature of the interface surface
S	Bounding surface (interface surface equation $S_1 = 0$; rigid surface equation $S_2 = 0$)
T	Surface tension
a	Radius of equilibrium interface surface or nondimensional depth of fluid layer with flat interface surface at $x = 0$ or $r = 0$
b	Half length of finite flat uniform fluid layer
g	Magnitude of body forces
h	Characteristic length of layers
i	$\sqrt{-1}$
k, ℓ, p, ℓ_n, p_n	Constants used in separation of variables
$\vec{k}$	Body force
m, n	Integers
$\vec{n}$	Unit vector normal to rigid surface S_2
$\bar{p}$	Pressure in fluid
p_e	Ambient pressure distribution at equilibrium interface surface

Notation (cont'd)

r, θ	Polar coordinates
r, z, θ	Cylindrical coordinates
r, θ, φ	Spherical coordinates
r_0	Nondimensional radius of axially symmetric flat interface, also nondimensional radius at $\theta = 0$ of the rigid supporting surface S_2 of layers with curved interface
t	Time
$\vec{v}$	Fluid velocity
x, z	Rectangular coordinates
$\bar{x}, \bar{z}$	Non-dimensional length parameters $\bar{x} = \frac{x}{h}$, $\bar{z} = \frac{z}{h}$
x_0	$2x_0$ = non-dimensional length of two-dimensional fluid layers with flat interface at $z = 0$
Φ	Velocity potential
Ψ	Potential function in terms of space variables
Ω	Potential of body forces
α	Largest angle subtended by layers with curved interface
α_n	Coefficient of infinite series
β	$\frac{\sigma h}{g}$ or $\frac{\sigma a}{g}$
ζ	First order small perturbation of interface surface
λ	Free parameter
μ	Stability parameter $\mu \equiv \frac{T}{\rho g h^2}$, or $\frac{T}{\rho g a^2}$
ρ	Density of fluid
σ	Frequency parameter (square of eigenfrequency)

I. INTRODUCTION

The significance of the phenomenon of interfacial instability of liquid layers in certain ablative heat protection problems, as well as in the behavior of fluids in containers under near-zero gravity conditions, has been repeatedly pointed out.[1-3] The theoretical analysis of this phenomenon has so far been largely restricted to layers of extremely simple geometry, in particular to layers with a uniform thickness.[4-11] In this investigation we consider layers with a non-uniform thickness and interface surfaces with a constant mean curvature. With the aid of the results of a recent study of the effects of curvature and unidirectional body forces on the stability of liquid layers[2] and the application of a semi-inverse method of finding exact solutions to the free oscillation problem of a fluid in a container[12] we derive stability criteria without resorting to approximate methods.

As in reference (2), the density of fluid separated by the interface from the layer under consideration is taken as zero while the fluid of the layer is assumed to be incompressible and inviscid and the flow irrotational. Moreover, we again define the equilibrium configuration of a liquid layer as stable whenever any initial disturbance of sufficiently small amplitudes and velocities leads to a motion in which the amplitudes and velocities remain arbitrarily small. Also we assume that the eigenfunctions corresponding to the free vibration problem are complete, which allows us to express the motion of the fluid due to any initial disturbance as a linear combination of the eigenvibrations. We therefore may deduce as the necessary and sufficient stability condition that all the eigenvalues (squares of the eigenfrequencies) be real and positive.

For layers of uniform thickness spread over various portions of the exterior or interior of cylinders and spheres the stability criteria could be presented in terms of inequalities involving the magnitude of either the unidirectional body force, g, or the nondimensional stability parameter

$$\mu = \frac{T}{\rho g a^2}$$

in which T denotes the surface tension, ρ the fluid density and a the radius of the equilibrium interface surface. With these layers it is consistently the first antisymmetric eigenvibration which leads to the stability criterion, since the corresponding eigenfrequency ceases first to be real when we vary the body force (or stability parameter) and start out with a value for which the equilibrium is stable. We anticipate that this fact also prevails in layers with a variable thickness and therefore limit ourselves to determining only the eigenfrequency of the first antisymmetric eigenvibration.

While exact analytical solutions to the small vibration problem of layers with a nonuniform thickness can in general not be obtained, it is nevertheless possible to determine rigorously one single eigenvibration in certain cases by means of a simple semi-inverse method.(12) By choosing to find the first antisymmetric free oscillation with the aid of this method and comparing the results for various layers, we can study the influence of the thickness variation on the stability.

II. BASIC EQUATIONS

We consider layers of an incompressible liquid which are bounded by a rigid wetted surface S_2 and an interface surface S_1. The interface S_1 separates the fluid of the layer under consideration from a neighboring gas or fluid of zero density. Any motion of the liquid is assumed as irrotational, which allows the velocity $\vec{v}$ to be expressed with the aid of a velocity potential Φ as

$$\vec{v} = -\nabla\Phi \tag{1}$$

Denoting the Laplacian by ∇^2, the equation of continuity has the form

$$\nabla^2\Phi = 0 \tag{2}$$

We consider the presence of a body force $\vec{k}$ per unit mass which can be derived from a potential Ω such that

$$\vec{K} = -\nabla\Omega \tag{3}$$

Assuming the fluid as inviscid and having a density ρ, a pressure $\bar{p}$, we may express the equilibrium in the sense of d'Alembert in the form

$$\nabla\left[\bar{p} + \rho\left(\Omega - \frac{\partial\Phi}{\partial t} + \frac{1}{2}v^2\right)\right] = 0 \tag{4}$$

Integrating Eq. (4) and denoting by $F(t)$ an arbitrary function of time we obtain Euler's equation

$$\bar{p} + \rho\left(\Omega - \frac{\partial\Phi}{\partial t} + \frac{1}{2}v^2\right) = F(t) \tag{5}$$

Without loss of generality we may include $F(t)$ in the time variance of Φ. In addition to this we may linearize (5) for small flow velocities by neglecting the term $(1/2)\rho v^2$, which reduces equation (5) to

$$\bar{p} = \rho\left(\frac{\partial\Phi}{\partial t} - \Omega\right) \tag{6}$$

The kinematic boundary condition of the fluid layer can be written in the form

$$\frac{DS}{Dt} = 0 \tag{7}$$

where

$$S = 0 \tag{8}$$

represents the equation of the bounding surface in question.

Denoting the surface tension by T and the external pressure by p_e, the dynamic equilibrium of the interface surface S_1 can be expressed as

$$\bar{p} - p_e = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \equiv 2TH \tag{9}$$

where R_1 and R_2 are the principal radii of curvature and H the mean curvature. (The radii of curvature are taken as positive if the center of curvature lies within the fluid.)

III. SEMI-INVERSE METHOD OF FINDING EXACT EIGENVIBRATIONS

In investigating the problem of the free oscillations of an incompressible fluid in an axially symmetric container, considering exclusively flat interface surfaces and neglecting the effects of surface tension, B. A. Troesch[12] made use of a semi-inverse method which yields a rigorous solution for one of the eigenvibrations. We apply the basic idea of this method to our fluid layers in the following manner.

Instead of prescribing the geometry of a layer completely, we merely select the shape of the interface surface in a suitable manner. For appropriately chosen solutions to Laplace's equation which contain sufficiently many free parameters it is possible to satisfy the interface boundary conditions in the presence of surface tension and body forces. By establishing the streamlines defined by such solutions and requiring the solid supporting surface to constitute a streamline, we are in a position to review rigorous vibration solutions corresponding to a class of liquid layers with the same interface and different solid supporting surfaces and thus different thickness distributions.

It should be noted that the selection of the shape of the interface together with the body forces and surface tension define a certain ambient pressure distribution, the feasibility of which can be dealt with separately.

IV. EIGENVIBRATIONS AND STABILITY OF TWO-DIMENSIONAL LAYERS WITH FLAT INTERFACE

4.1. Rigorous Solutions to Free Vibration Problem

We refer the layer to Cartesian coordinates (x, z) such that for static equilibrium the interface surface is characterized by $z = 0$ (See Fig. 1). During the motion resulting from a small disturbance the interface surface S_1 deviates from its equilibrium position by a small distance $\zeta(x, t)$ and the equation for S_1 assumes the form

$$S_1(x, z, t) = z - \zeta(x, t) = 0 \tag{10}$$

With Eq. (10) the mean curvature H of the interface surface is given by

$$2H = \frac{1}{R_2(x, t)} = - \frac{\dfrac{\partial^2\zeta}{\partial x^2}}{\left[1 + \left(\dfrac{\partial\zeta}{\partial x}\right)^2\right]^{3/2}} \tag{11}$$

We assume that $\zeta(x, t)$ as well as its derivatives are of first order small and neglect all terms which are of second or higher order small. Thus Eq. (11) reduces to

$$2H = - \frac{\partial^2\zeta(x, t)}{\partial x^2} \tag{12}$$

We shall restrict ourselves for the present to unidirectional body forces with the potential

$$\Omega = gz \tag{13}$$

Substituting Eqs. (12), (13) and (6) into the dynamic boundary condition (9) we obtain

$$\rho\left[\frac{\partial\Phi}{\partial t}(x, z, t) - gz\right] - p_e(x) = -T\,\frac{\partial^2\zeta(x, t)}{\partial x^2} \tag{14}$$

With Eqs. (1) and (10) the kinematic boundary condition (7) can be written in linearized form as

$$\frac{\partial \Phi}{\partial z} = - \frac{\partial \zeta}{\partial t} \tag{15}$$

By differentiating (14) with respect to time and making use of (15) we can combine the kinematic and dynamic boundary conditions at the interface surface into a relation for the velocity potential

$$\rho \left(\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} \right) = T \frac{\partial^3 \Phi}{\partial x^2 \partial z}$$

With the aid of Eq. (2) we can write this combined boundary condition also in the form

$$\rho \left(\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} \right) + T \frac{\partial^3 \Phi}{\partial z^3} = 0 \tag{16}$$

Neglecting consistently terms which are of second and higher order small, we may require (16) to be satisfied at the equilibrium interface $z = 0$, rather than at the disturbed interface $z = \zeta(x, t)$.

Since we are seeking solutions in the form of harmonic oscillations we separate the space and time variables according to

$$\Phi(x, z, t) = \psi(x, z) e^{i\sqrt{\sigma}\, t} \tag{17}$$

and thereby reduce the initial-boundary value to an eigenvalue problem with $\psi(x, z)$ as eigenfunction and σ as eigenvalue. By substituting (17) into (2) and (16) we find

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{18}$$

and

$$\left[\rho \left(-\sigma \psi + g \frac{\partial \psi}{\partial z} \right) + T \frac{\partial^3 \psi}{\partial z^3} \right]_{z=0} = 0 \tag{19}$$

We note that the eigenvalue problem for ψ is of a special type since the eigenvalue appears exclusively in the boundary condition and not in the differential equation and since the boundary condition (19) contains derivatives of higher order than does the differential equation. With the transformations

$$\bar{x} = \frac{x}{h}, \quad \bar{z} = \frac{z}{h}$$

where h is a characteristic length of the layer, we can rewrite Eqs. (18) and (19) in terms of nondimensional space parameters

$$\frac{\partial^2 \psi}{\partial \bar{x}^2} + \frac{\partial^2 \psi}{\partial \bar{z}^2} = 0 \tag{18'}$$

$$\left[-\beta\psi + \frac{\partial \psi}{\partial \bar{z}} + \mu \frac{\partial^3 \psi}{\partial \bar{z}^3} \right]_{\bar{z}=0} = 0 \tag{19'}$$

where

$$\beta = \frac{\sigma h}{g} \quad \text{and} \quad \mu = \frac{T}{\rho g h^2} \tag{20}$$

For the sake of convenience we replace again $\bar{x}$ by x and $\bar{z}$ by z with the understanding that x and z are now nondimensional length parameters. The eigenvalue problem can thus be presented in the form

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{18}$$

$$\left[-\beta\psi + \frac{\partial \psi}{\partial z} + \mu \frac{\partial^3 \psi}{\partial z^3} \right]_{z=0} = 0 \tag{19''}$$

We consider now the class of exact solutions for ψ which can be obtained by separating the space variables:

$$\psi(x, z) = X(x) \cdot Z(z) \tag{21}$$

By substituting (21) into (18) we deduce

$$\frac{X''}{X} = - \frac{Z''}{Z} = k^2 \tag{22}$$

whereby k^2 may assume positive as well as negative values. We consider first $k \neq 0$. According to (22) ψ must be of the form

$$\psi(x,z) = [A \sinh(kx) + B \cosh(kx)]\ [C \sin(kz) + \cos(kz)] \tag{23}$$

if $k \neq 0$. By requiring expression (23) to satisfy the interface boundary condition (19") we obtain

$$-\beta + k(1 - \mu k^2)C = 0 \tag{24}$$

Since the solid supporting surface S_2 of the layer must constitute a streamline, S_2 is given by

$$\nabla\psi\cdot\vec{n} = 0 \tag{25}$$

or

$$\frac{\partial\psi}{\partial x}\, dz - \frac{\partial\psi}{\partial z}\, dx = 0 \tag{26}$$

where $\vec{n}$ is the normal vector of the surface. Integrating (26) by making use of Eqs. (21), (22) and (23), we find

$$\begin{aligned} S_2(x,z) &= X'(x)\cdot Z'(z) - E \\ &= k^2[A \cosh(kx) + B \sinh(kx)]\cdot[C \cos(kz) - \sin(kz)] - E = 0 \end{aligned} \tag{27}$$

where E is a constant.

If we restrict ourselves to layers which are symmetric with respect to the z-axis, we have

$$B = 0$$

Hence the eigenfunction and the solid surface S_2 are given by

$$\psi(x,z) = A \sinh(kx)\ [C \sin(kz) + \cos(kz)] \tag{28}$$

and

$$S_2(x,z) = \cosh(kx)[C \cos(kz) - \sin(kz)] - E = 0 \tag{29}$$

where E is a new constant. We specifically consider the following three cases:

(a) $E = 0$

In this case the layer is of uniform depth and bounded by the solid surfaces $x = \pm x_0$ and $z = -a$ as illustrated in Fig. 2. The corresponding value for k is purely imaginary

$$k = i\ell \tag{30}$$

and

$$x_0 = \frac{2n - 1}{2\ell}\pi, \quad \ell > 0, \quad n = \text{integer} \tag{31}$$

$$C = -i \tanh(\ell a) \tag{32}$$

Substituting (30) and (32) into (24) and (28) we obtain a simple relation for the eigenvalue σ and the corresponding eigenfunction:

$$\beta = \frac{\sigma h}{g} = \ell(1 + \mu\ell^2) \tanh(\ell a) \tag{33}$$

$$\Psi(x, z) = D \sin(\ell x) \cosh[\ell(a + z)] \tag{34}$$

(b) $E \neq 0$

The parameters C and E in Eqs. (29) may be determined for example by requiring the surface S_2 to pass through the points $(0, -a)$ and $(x_0, 0)$. Thus we find for an arbitrary k (real or imaginary)

$$\beta = \frac{\sigma h}{g} = k(1 - \mu k^2) \frac{\sin(ka)}{\cosh(kx_0) - \cos(ka)} \tag{35}$$

$$S_2(x,z) = \cosh(kx)\left[\cos(kz) - \frac{\cosh(kx_0) - \cos(ka)}{\sin(ka)} \sin(kz)\right] - \cosh(kx_0) = 0$$

$$\Psi(x,z) = D \sinh(kx) [\cos k(a + z) - \cos(kz)\cosh(kx_0)] \tag{36}$$

Representative layers corresponding to various values of k are shown in Fig. 3.

(c) $k = 0$

In this case the solution to (18) is of the form

$$\Psi(x, z) = (Ax + B)(Cz + D) \tag{37}$$

and the interface boundary condition (19") is satisfied if

$$C = \beta D \tag{38}$$

Considering only layers which are symmetric with respect to $x = 0$, we require

$$B = 0 \tag{39}$$

The integration of (26) leads to the expression

$$S_2(x,z) = \left(z + \frac{D}{C}\right)^2 - x^2 - E = 0 \tag{40}$$

Choosing the values of (D/C) and E such that the surface S_2 contains the points $(0, -a)$, $(x_0, 0)$ we obtain

$$\frac{C}{D} = \beta = \frac{2a}{a^2 + x_0^2} \tag{41}$$

and layers as illustrated in Fig. 4.

4.2. Stability Criteria

Assuming that the liquid layer under consideration has finite dimensions and a complete set of eigenfunctions $\Psi_1, \Psi_2, \ldots, \Psi_n, \ldots$, we may express the motion of the fluid due to any initial disturbance in the form

$$\Phi = \sum_{n=1,2,\ldots} \alpha_n \Psi_n \; e^{i\sqrt{\sigma_n}\, t} \tag{42}$$

from which we immediately derive

$$\sigma_n \text{ real}, \quad \sigma_n > 0, \quad n = 1, 2, \ldots \tag{43}$$

as the necessary and sufficient stability criterion.

We apply this criterion to the eigenvibration obtained in a rigorous manner in the previous section and we consider again the three different cases: $E = 0$, $E \neq 0$, and $k = 0$.

(a) $E = 0$

Requiring $\sigma > 0$ in (33), we deduce

$$g(1 + \mu \ell^2) > 0 \tag{44}$$

With Eqs. (20), (31) and

$$x_0 = \frac{b}{h} \tag{45}$$

where $2b$ is the width of the layer [See Fig. 2] we can reduce (44) to the classical result

$$\frac{\rho g}{T} > - \frac{\pi^2}{4b^2} \tag{46}$$

(b) $E \neq 0$

From Eq. (35) follows

$$\sigma > 0 \quad \text{if} \quad gk(1 - \mu k^2)\sin(ka) > 0 \tag{47}$$

We note that Eqs. (35) and (36) do not change if we replace k by $-k$ and may therefore restrict ourselves to positive values of k. Since the slope of the solid surface S_2 at $x = x_0$, $z = 0$ is given by

$$\left.\frac{dz}{dx}\right|_{\substack{x=x_0 \\ z=0}} = \frac{\sinh(kx_0)\sinh(ka)}{\cosh(kx_0)[\cosh(kx_0) - \cos(ka)]}$$

its sign is the same as that of $\sin(ka)$.

This means that we have a convex solid surface for $\sin(ka) < 0$ and a concave one for $\sin(ka) > 0$. Hence considering positive as well as negative values for g we may restrict ourselves to $\sin(ka) > 0$ without loss of generality. We obtain as stability condition

$$\sigma > 0 \quad \text{if} \quad \frac{\rho g}{T} > \frac{k^2}{h^2} \; [\sin(ka) > 0] \tag{48}$$

Similarly, we deduce for $k = i\ell$

$$\sigma > 0 \quad \text{if} \quad \frac{\rho g}{T} > - \frac{\ell^2}{h^2} \tag{49}$$

(c) $k = 0$

In this case it immediately follows from Eq. (41) that

$$\sigma > 0 \quad \text{if} \quad g > 0 \tag{50}$$

For a real k the layers are only stable if the body forces exceed a certain positive limit value while for an imaginary k the body forces may come close to a certain negative limit value before instability is reached. We conclude from this that the variation of the thickness of a liquid layer may significantly influence the stability criterion. Besides this, we may utilize this fact in designing containers of fluids for near-zero gravity conditions in such a manner that the fluid adheres to a certain area of the container.

V. EIGENVIBRATIONS AND STABILITY CRITERIA OF THREE-DIMENSIONAL LAYERS WITH FLAT INTERFACE

As illustrated in Fig. 5, we refer the layer to cylinder coordinates z, r, θ such that for equilibrium the interface surface S_1 is defined by $z = 0$. Denoting the deviation of the interface from its equilibrium position by $\zeta(r, \theta, t)$ we may write for the surface S_1

$$S_1(z, r, \theta, t) = z - \zeta(r, \theta, t) = 0 \tag{51}$$

and its mean curvature in linearized form

$$2H = -\left(\frac{\partial^2\zeta}{\partial r^2} + \frac{1}{r}\frac{\partial\zeta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\zeta}{\partial\theta^2}\right) \tag{52}$$

As before, we assume the presence of body forces whose potential is defined by Eq. (13). The combination of the dynamic and kinematic boundary conditions leads to

$$\left[\rho\left(\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z}\right) + T\frac{\partial^3\phi}{\partial z^3}\right]_{z=0} = 0 \tag{53}$$

We separate the time and space variables according to

$$\Phi(z, r, \theta, t) = \Psi(z, r, \theta)e^{i\sqrt{\sigma}\,t} \tag{54}$$

If we refer the length parameters again to a characteristic length h we may interpret z and r as nondimensional quantities and define the eigenvalue problem for Ψ by

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{\partial^2\Psi}{\partial z^2} = 0 \tag{55}$$

$$\left[\beta\Psi - \frac{\partial\Psi}{\partial z} - \mu\frac{\partial^3\Psi}{\partial z^3}\right]_{z=0} = 0 \tag{56}$$

Exact solutions for Ψ can be obtained in the form

$$\Psi(z, r, \theta) = F(z, r)e^{im\theta} \tag{57}$$

which requires F to satisfy the equations

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} - \frac{m^2}{r^2}F + \frac{\partial^2 F}{\partial z^2} = 0 \tag{58}$$

$$\left[\beta F - \frac{\partial F}{\partial z} - \mu\frac{\partial^3 F}{\partial z^3}\right]_{z=0} = 0 \tag{59}$$

Considering first such functions which permit an additional separation of variables, we find

$$F(z, r) = A[\cosh(kz) + B\sinh(kz)]J_m(kr) \tag{60}$$

where $J_m(kr)$ denotes the m^{th} order Bessel function of the first kind and k the separation parameter. The substitution of (60) into (59) leads to

$$\beta = \frac{\sigma h}{g} = k(1 + \mu k^2)B$$

or

$$\sigma = \frac{g}{h}k(1 + \mu k^2)B \tag{61}$$

The stability conditions are

$$k > 0, \quad B \gtrless 0, \quad \frac{\rho g}{T} \gtrless -\frac{k^2}{h^2} \tag{62}$$

and the corresponding equation for S_2

$$k > 0,\ m > 0,\ \ln[\sinh(kz) + B\cosh(kz)] - k^2\int_0^r \frac{J_m(kr)}{\frac{\partial J_m(kr)}{\partial r}}\,dr + C = 0 \tag{63}$$

Layers with solid surfaces defined by (63) are illustrated in Figure 6.

We consider the following special cases

(a) $F_z = 0, \ F_r = 0$

Requiring S_2 to contain the points $(0, -a)$ and $(r_0, 0)$ we have in this case a layer of uniform thickness bounded by the surface $z = -a$ and the cylinder $r = r_0$ (See Fig. 7). The relation for the eigenfrequencies corresponding to mode shapes as defined by (60) is

$$\sigma = \frac{g}{h} k(1 + \mu k^2)\tanh(ka) \tag{64}$$

where the possible values of k are given by

$$J_m'(kr_0) = 0 \qquad m = 1, 2, \ldots \tag{65}$$

From (64) we find the stability condition

$$\frac{\rho g}{T} > - \frac{k^2}{h^2} \tag{66}$$

which indicates that the lowest value for k is most significant: $m = 1$, $kr_0 = 1.85$.

(b) $k = 0$

In this case the velocity potential can be written in the form

$$\Psi(r, z, \theta) = A\, r^m (z + B)e^{im\theta}, \quad m \neq 0 \tag{67}$$

The interface boundary conditon (59) yields

$$B = \frac{1}{\beta} \tag{68}$$

and the equation of the solid surface S_2 becomes

$$S_2 = m\left(\frac{1}{2} z^2 + Bz\right) - \frac{1}{2} r^2 + C = 0 \tag{69}$$

By requiring the surface S_2 to pass through the points $(r = 0, z = -a)$ and $(r = r_0, z = 0)$ we find

$$B = \frac{a^2 m + r_0^2}{2ma} \tag{70}$$

and

$$S_2 = mz^2 + \frac{a^2m + r_0^2}{a} z + r_C^2 - r^2 = 0 \tag{71}$$

From (68) and (70) we obtain as stability condition

$$g > 0 \tag{72}$$

which holds independently of m, r_0 and a. Representative layers defined by (71) are shown in Fig. 8.

As in the two-dimensional case, we conclude that the variation of the thickness can markedly influence the stability of three-dimensional layers with a flat interface, particularly for body forces in the near-zero region. We also note that for some solid supporting surfaces S_2 (container walls) layers of this type are stable if the body forces are larger than a certain negative limit value while for other surfaces S_2 the body forces have to be positive for stability. Moreover, we may again suggest the feasibility of exploiting such a behavior in the design of fluid containers for near-zero gravity conditions.

VI. ON THE STABILITY OF LAYERS WITH UNIFORMLY CURVED INTERFACE

6.1. Two-Dimensional Layers with Curved Interface

We refer the layer to cylindrical coordinates such that the equilibrium configuration is characterized by $r = a$ as shown in Figure 9. In the presence of a disturbance the equation for the interface surface may be written in the form

$$S_1(r, \theta, t) = r - [a + \zeta(\theta, t)] = 0 \tag{73}$$

In terms of cylindrical coordinates, the uniaxial body forces are now defined by the potential

$$\Omega = gr \cos \theta \tag{74}$$

and the linearized mean curvature given by

$$2H = \frac{1}{R_2} = \frac{1}{a} \left[1 - \frac{\zeta(\theta,t)}{a} - \frac{1}{a} \frac{\partial^2}{\partial \theta^2} \zeta(\theta,t) \right] \tag{75}$$

By combining the dynamic and kinematic boundary conditions as before, we obtain

$$\rho \left[\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial r} \cos \theta \right]_{r=a} = \frac{T}{a^2} \left[\frac{\partial \Phi}{\partial r} + \frac{\partial^3 \Phi}{\partial r \partial \theta^2} \right]_{r=a} \tag{76}$$

We again separate the time and space variables according to

$$\Phi(r,\theta,t) = \Psi(r,\theta) e^{i\sqrt{\sigma}\, t} \tag{77}$$

and may interpret r as nondimensional length parameter by using "a" as the reference length. Thus we arrive at the eigenvalue problem for Ψ defined by

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} = 0 \tag{78}$$

$$\left[\beta\Psi - \frac{\partial\Psi}{\partial r}\cos\theta + \mu\left(\frac{\partial\Psi}{\partial r} + \frac{\partial^3\Psi}{\partial r\partial\theta^2}\right)\right]_{r=1} = 0 \tag{79}$$

where β and μ now abbreviate

$$\mu = \frac{T}{\rho g a^2} \quad \text{and} \quad \beta = \frac{\sigma a}{g} \tag{80}$$

The streamlines can be obtained by integrating the relation

$$r\frac{\partial\Psi}{\partial r}\, d\theta = \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\, dr \tag{81}$$

Any function of the form

$$\Psi(r,\theta) = \sum_{n=1}^{N} (r^{\ell_n} + A_n r^{-\ell_n})[B_n \sin(\ell_n\theta) + D_n \cos(\ell_n\theta)] \tag{82}$$

with arbitrary ℓ_n and N represents a solution to (78). In particular, we may choose $\ell_n = n$ and $N = 3$:

$$\Psi(r,\theta) = \sum_{n=1}^{3} (r^n + A_n r^{-n})[B_n \sin(n\theta) + D_n \cos(n\theta)] \tag{83}$$

We require $D_n = 0$, $n = 1,2,3$ and thus restrict our attention again to the antisymmetric modes of oscillation. The substitution of

$$\Psi(r,\theta) = \sum_{n=1}^{3} B_n[r^n + A_n r^{-n}]\sin(n\theta)$$

into the interface boundary condition (79) leads to the relations

$$\left.\begin{aligned}
&B_3/B_2 = \lambda \\
&A_2 = 1 - 2\beta\lambda \\
&B_1/B_2 = \lambda + 2\beta[1 - \lambda(\beta + 6\mu)] \\
&A_1 B_1/B_2 = \lambda - 2\beta[1 - \lambda(\beta + 6\mu)] \\
&A_3 = 1
\end{aligned}\right\} \qquad (84)$$

in which λ is an arbitrary parameter. If we select

$$\lambda = \frac{1}{\beta + 6\mu} \qquad (85)$$

we limit our analysis to a family of layers for which we can readily derive a stability condition by replacing the abbreviations β and μ in (85) by their original form and requiring σ to be positive. We deduce

$$\begin{aligned}
&a > 0, \quad g > \frac{6\lambda T}{\rho a^2} \\
&a < 0, \quad g < \frac{6\lambda T}{\rho a^2}
\end{aligned} \qquad (86)$$

With (85) the differential equation for the stream lines becomes

$$\frac{dr}{d\theta} = \frac{r\{\lambda(r - r^{-1})\sin\theta + 2[r^2 - (1 - 2\lambda\beta)r^{-2}]\sin 2\theta + 3\lambda(r^3 - r^{-3})\sin 3\theta\}}{\lambda(r + r^{-1})\cos\theta + 2[r^2 + (1 - 2\lambda\beta)r^{-2}]\cos 2\theta + 3\lambda(r^3 + r^{-3})\cos 3\theta} \qquad (87)$$

Representative layers with such streamlines are illustrated in Figure 10.

6.2. Two-Dimensional Layers with Slightly Curved Interface

For such layers we may write

$$\cos\theta \approx 1 \qquad (88)$$

and as interface condition

$$\left[\beta\psi - \frac{\partial\psi}{\partial r} + \mu\left(\frac{\partial\psi}{\partial r} + \frac{\partial^3\psi}{\partial r\partial\theta^2}\right)\right]_{r=1} = 0 \tag{89}$$

This condition is satisfied by

$$\psi(r,\theta) = B(r^{\ell} + Ar^{-\ell})\sin \ell\theta \tag{90}$$

for arbitrary ℓ if

$$A = \frac{\ell[1 + \mu(\ell^2 - 1)] - \beta}{\ell[1 + \mu(\ell^2 - 1)] + \beta} \tag{91}$$

The corresponding stream lines are given by

$$B[r^{\ell} - Ar^{-\ell}]\cos(\ell\theta) - C = 0 \tag{92}$$

where C denotes a constant. By choosing $C = 0$ we arrive at a family of layers with uniform thickness and the same stability criteria as discussed in Ref. 2. For $C \neq 0$ we may require the solid boundary to contain two prescribed points, e.g.

$$\theta = 0, \quad r = r_0; \quad \theta = \pm\alpha, \quad r = 1$$

which determines A and C/B in (92):

$$A = \frac{r_0^{\ell} - \cos(\ell\alpha)}{r_0^{-\ell} - \cos(\ell\alpha)} \tag{93}$$

$$\frac{C}{B} = \frac{r_0^{-\ell} - r_0^{\ell}}{r_0^{-\ell} - \cos(\ell\alpha)}\cos(\ell\alpha) \tag{94}$$

By comparing (93) and (91) we find the relation

$$\sigma = \frac{\ell}{a}\,\frac{1 - r_0^{2\ell}}{1 + r_0^{2\ell} - 2r_0^{\ell}\cos(\ell\alpha)}\left[g + \frac{T}{\rho a^2}(\ell^2 - 1)\right] \tag{95}$$

from which we can deduce the stability conditions

$$a > 0 \quad r_0 < 1: \quad \frac{\rho g a^2}{T} > 1 - \ell^2$$

$$a < 0 \quad r_0 > 1: \quad \frac{\rho g a^2}{T} < 1 - \ell^2$$

As a special case we consider $\ell = 1$ and $A = 0$. The substitution of (93) and (94) into (92) yields for the solid boundary

$$r \cos \theta = r_0 \tag{96}$$

i.e., a flat surface. The corresponding velocity potential is given by

$$\Psi = Br \sin \theta \tag{97}$$

Specific examples of two-dimensional layers with a slightly curved interface are illustrated in Fig. 11.

6.3. Three-Dimensional Layers with Curved Interface

In terms of spherical coordinates we may define the disturbed interface by

$$S_1(r, \theta, \varphi, t) = r - [a + \zeta(\theta, \varphi, t)] = 0 \tag{98}$$

where $r = a$ represents the equilibrium configuration as shown in Fig. 12. The potential of uniaxial body forces is again given by (74). For small ζ the mean curvature may be written in the form[13]

$$2H = \frac{1}{a}\left[2 - \frac{1}{a}\left(2\zeta + \frac{\partial\zeta}{\partial\theta}\cot\theta + \frac{\partial^2\zeta}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2\zeta}{\partial\varphi^2}\right)\right] \tag{99}$$

A combination of the kinematic and dynamic boundary conditions for the interface leads to

$$\left[\rho\left(\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial r}\cos\theta\right) - \frac{T}{a^2}\left(2\frac{\partial\Phi}{\partial r} + \cot\theta\frac{\partial^2\Phi}{\partial r\partial\theta} + \frac{\partial^3\Phi}{\partial r\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^3\Phi}{\partial r\partial\varphi^2}\right)\right]_{r=a} = 0 \tag{100}$$

The physical nature of the problem suggests a separation of the variables according to

$$\Phi(r,\ \theta,\ \varphi,\ t) = \Psi(r,\ \theta)e^{im\varphi}\ e^{i\sqrt{\sigma}\ t} \tag{101}$$

in which m denotes an integer. We may again interpret r as the nondimensional length parameter, using the radius "a" of the equilibrium shape as reference length. The eigenvalue problem for Ψ is now defined by

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{\sin\theta}\ \frac{\partial}{\partial \theta}\ \left(\sin\theta\, \frac{\partial \psi}{\partial \theta}\right) - \frac{m^2}{\sin^2\theta}\ \psi = 0 \tag{102}$$

and

$$\left[\beta\psi - \cos\theta\, \frac{\partial \psi}{\partial r} - \mu\left(4r\, \frac{\partial^2 \psi}{\partial r^2} + r^2\, \frac{\partial^3 \psi}{\partial r^3}\right)\right]_{r=1} = 0 \tag{103}$$

The nondimensional quantities β and μ are again defined by (80) and the stream lines by (81). Exact solutions to the eigenvalue problems are for example[14].

$$\psi(r,\theta) = \sum_n A_n[r^{p_n} + B_n r^{-(p_n+1)}]P^m_{p_n}(\cos\theta) \tag{104}$$

in which $P^m_{p_n}$ represents the associated Legendre function of degree m and order p_n [15]. The order p_n is arbitrary while the degree m is integer.

We consider a particular family of layers to which (104) represents an exact solution by selecting

$$m = 1\ , \qquad p_n = 1,\ 2$$

or

$$\Psi = A_1[(r + B_1 r^{-2})\sin\theta + A_2(r^2 + B_2 r^{-3})\sin\theta\cos\theta] \tag{105}$$

This function satisfies the interface condition if

$$B_1 = -1, \quad B_2 = \frac{2}{3}, \quad A_2 = \frac{9}{5\beta} \tag{106}$$

From (106) and (80) follows

$$\sigma = \frac{9}{5A_2} \frac{g}{a} \tag{107}$$

For $A_2 > 0$, $a > 0$ we find as the stability condition

$$g > 0$$

Layers representative for this family are shown in Fig. 13.

6.4. Three-Dimensional Layers with Slightly Curved Interface

For such layers we have

$$\cos\theta \approx 1, \quad \sin\theta \simeq \theta \tag{108}$$

and therefore as the continuity equation and the interface condition

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial\psi}{\partial r}\right) + \frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial\psi}{\partial\theta}\right) - \frac{m^2}{\theta^2}\psi = 0 \tag{109}$$

$$\left[\beta\psi - \frac{\partial\psi}{\partial r} - \mu\left(4r\frac{\partial^2\psi}{\partial r^2} + r^2\frac{\partial^3\psi}{\partial r^3}\right)\right]_{r=1} = 0 \tag{110}$$

Solutions to Eq. (109) which satisfy the interface condition (110) can be written in the form

$$\psi(r,\theta) = [r^p + Ar^{-(p+1)}]J_m[\sqrt{p(p+1)}\,\theta] \tag{111}$$

where

$$A = \frac{-\beta + p\{1 + \mu[p(p+1) - 2]\}}{+\beta + (p+1)\{1 + \mu[p(p+1) - 2]\}} \tag{112}$$

and

$$p > 0 .$$

The streamlines corresponding to (111) are

$$\ln\left[r^{(p+1)} - A\left(\frac{p+1}{p}\right) r^{-p}\right] = \int_0^{\sqrt{p(p+1)}\,\theta} \frac{J_m(\lambda)}{J_m'(\lambda)}\, d\lambda + C \quad (m > 0) \tag{113}$$

with C denoting a constant.

For C = 0 we have a family of three-dimensional layers of uniform thickness as illustrated in Fig. 14. By defining the solid surface as

$$r = r_0, \quad \theta = \pm \alpha \quad (\alpha \text{ small}) \tag{114}$$

and requiring

$$\frac{\partial\psi}{\partial r} = \frac{\partial\psi}{\partial\theta} = 0$$

we obtain

$$A = \frac{p}{p+1} \; r_0^{2p+1} \tag{115}$$

and

$$J_m' \left[\sqrt{p(p+1)}\,\alpha\right] = 0 \tag{116}$$

The comparison of (115) with (112) leads to

$$\sigma = \frac{1}{a} \; \frac{p(1 - r_0^{2p+1})}{1 + \frac{p}{p+1} r_0^{2p+1}} \left\{ g + \frac{T}{\rho a^2} \; [p(p+1) - 2] \right\} \tag{117}$$

with $r_0 < 1$, and $p > 0$, we have

$$g + \frac{T}{\rho a^2} \; [p(p+1) - 2] > 0$$

and therefore as stability condition

$$g > [2 - p(p+1)] \frac{T}{\rho a^2} \tag{118}$$

Similarly, we find for $r_0 > 1$

$$g < [2 - p(p+1)] \frac{T}{\rho a^2} \tag{119}$$

It should be noted that for a given α the parameter p is determined by (116). According to (118) and (119) we obtain the stability criterion with the smallest value of p. The latter is given by the smallest root of

$$J_1' \left[\sqrt{p(p+1)}\, \alpha\right] = 0 \tag{120}$$

The physical interpretation of this fact is that among all eigenvibrations the first antisymmetric oscillation leads to the stability criterion since the corresponding frequency approaches zero first in varying the parameters T, ρ and g.

The stability criterion for an axially symmetric layer with a flat interface, a uniform thickness and a radius b as shown in Fig. 7 is

$$g > -13.7 \frac{T}{\rho (2b)^2}$$

while that of the slightly curved layer of uniform thickness and the same interface area (see Fig. 14) is

$$g > \left[\frac{2}{a^2} - \frac{13.7}{(2b)^2}\right] \cdot \frac{T}{\rho}$$

As in the two-dimensional case[2] we observe that the curvature has a destabilizing effect.

If $\underline{C \neq 0}$ in Eq. (113), we can interpret (112) as the frequency equation for arbitrary values of p and A:

$$\sigma = \frac{1}{a} \frac{[p - (p+1)A]}{(1+A)} \left\{ g + \frac{T}{\rho a^2} [p(p+1) - 2] \right\} \tag{121}$$

For stability we deduce

$$\frac{p - (p+1)A}{1 + A} > 0 \ , \quad a \gtrless 0 \ , \quad g \gtrless \frac{T}{\rho a^2} [2 - p(p+1)]$$

$$\frac{p - (p+1)A}{1+A} < 0 , \quad a \gtrless 0 , \quad g \lessgtr \frac{T}{\rho a^2} \, [2 - p(p+1)] \tag{122}$$

Representative examples of slightly curved three-dimensional layers are illustrated in Fig. 15.

If we assume $p(p+1)\theta \ll 1$, we may obtain approximate solutions to the streamline Eq.(113) in the form

$$\int \frac{J_m(\lambda)}{J_m'(\lambda)} \, d\lambda \simeq -(m+1)\ln\left[1 - \frac{\lambda^2}{2m(m+1)}\right]$$

For $m = 1$ we have

$$\int_0^{\sqrt{p(p+1)}\,\theta} \frac{J_m(\lambda)}{J_m'(\lambda)} \, d\lambda = -2 \ln\left[1 - \frac{p(p+1)}{4}\theta^2\right]$$

and thus

$$S_2 = \left[r^{p+1} - A \, \frac{p+1}{p} \, r^{-p}\right] \left[1 - \frac{p(p+1)}{4}\theta^2\right]^2 - C = 0 \tag{123}$$

Requiring the solid surface to contain the points $(r = 1, \;\; \theta = \alpha)$ and $(r = r_0, \;\; \theta = 0)$ we find

$$A \simeq \frac{p\left\{r_0^{p+1} - \left[1 - \frac{p(p+1)}{4}\alpha^2\right]^2\right\}}{(p+1)\left\{r_0^{-p} - \left[1 - \frac{p(p+1)}{4}\alpha^2\right]^2\right\}} \tag{124}$$

and

$$S_2 \simeq \left(r^{p+1}\left\{r_0^{-p} - \left[1 - \frac{p(p+1)}{4}\alpha^2\right]^2\right\} - r^{-p}\left\{r_0^{p+1} - \left[1 - \frac{p(p+1)}{4}\alpha^2\right]^2\right\}\right) \cdot$$

$$\cdot \left[1 - \frac{p(p+1)}{4}\theta^2\right]^2 - (r_0^{-p} - r_0^{p+1})\left[1 - \frac{p(p+1)}{4}\alpha^2\right]^2 = 0 \tag{125}$$

For $p = 1$ and $A = 0$ we have an approximation to the important case of a layer on a flat surface

$r \cos \theta = r_0$

as shown in Figure 15b.

VII. CONCLUSION

The results of this study clearly indicate a strong dependence of the stability of all layers (two- and three-dimensional) on the thickness variation, especially in the realm of near-zero gravity conditions. Of particular significance is the fact that for certain solid supporting surfaces (container walls) the fluid layers are stable only if the unidirectional body force is either positive or exceeds a certain positive value, while for other surfaces stability prevails for limited negative values of the body force. By taking advantage of this fact in the design of containers of fluid for near-zero gravity conditions, it appears possible to enforce the permanent adherence of the fluid to a certain area of the container.

It should be noted that we have not introduced any contact angle considerations in explicit form in our analysis, and therefore we have to require that the dimensions of the interface surface be large compared with the effective range of the contact forces. Also, it should be pointed out that our stability criteria are conservative only if the "appropriately chosen" solution to Laplace's equation actually leads to the fundamental eigenvibration* whose eigenfrequency ceases first to be real in varying the body force or stability parameter. Partial reassurance in this regard is offered by the fact that our results include as special cases the known stability criteria for layers of uniform thickness with either flat or uniformly curved interface surfaces.

*Vibration with the lowest eigenfrequency.

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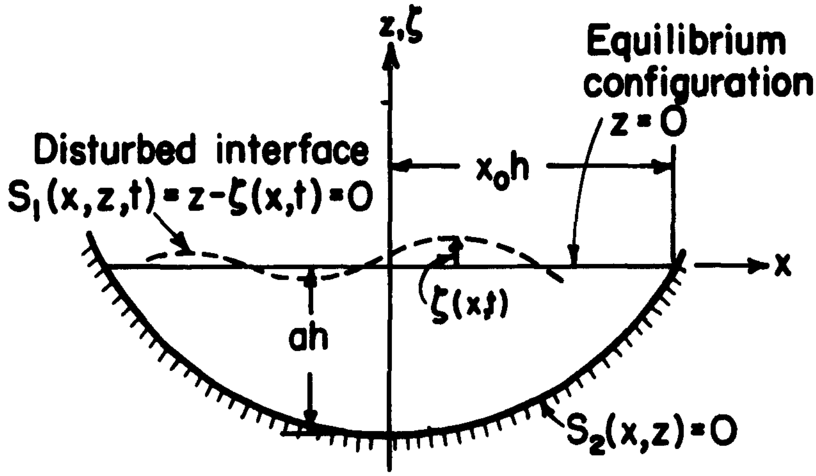


Fig. 1. Reference Frame for Two-Dimensional Layer with Flat Interface.

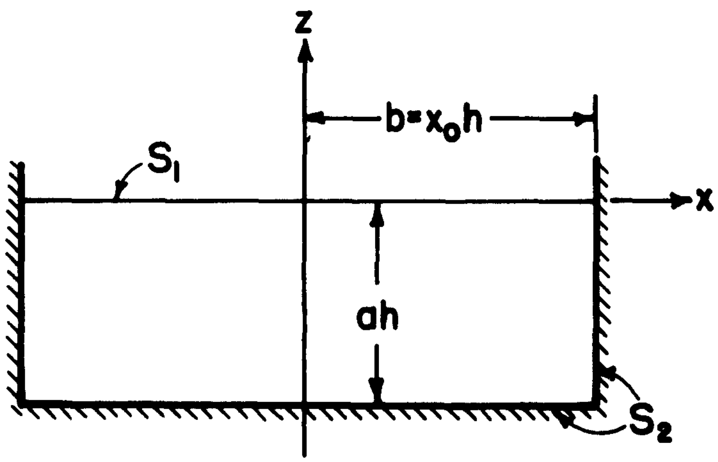


Fig. 2 Two-Dimensional Layer whose Free Oscillations have the Velocity Potential $\Phi(x,z,t) = D\sin(lx)\cosh[l(a+z)]\cdot e^{i\sqrt{\sigma}t}$ where $l = [(2n-1)/2]\pi(h/b)$.

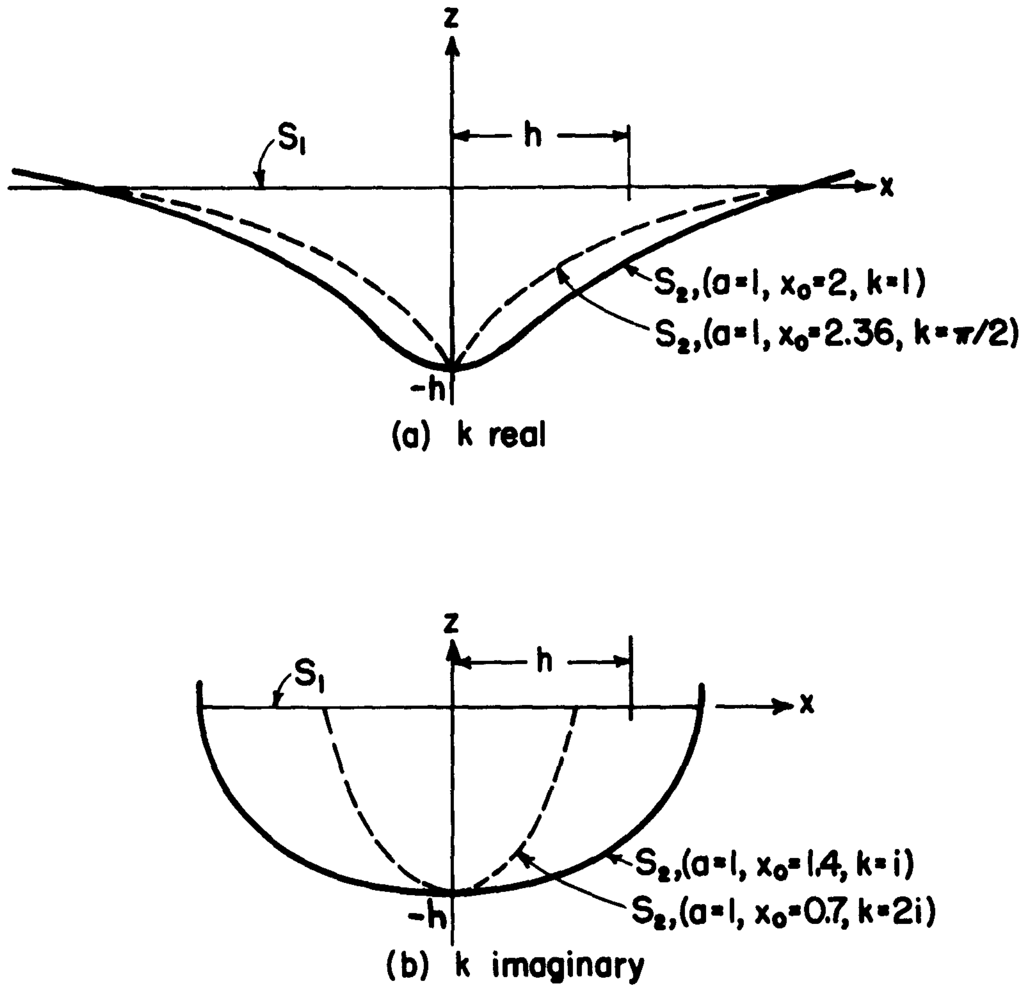


Fig. 3 Two-Dimensional Layers with Free Oscillations Defined by $\Phi(x,z,t) = D\sinh(kx)[\cos k(a+z) - \cos(kz)\cosh(kx_0)]e^{i\sqrt{\sigma}t}$; (a) k real, (b) k imaginary.

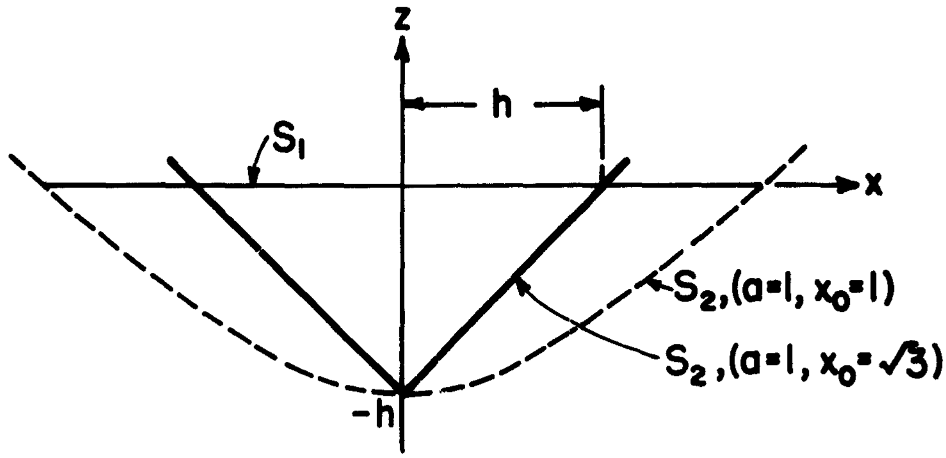


Fig. 4. Two-Dimensional Layers with Free Oscillations Defined by $\Phi(x,z,t) = Ax\{1+[2a/(a^2+x_0^2)]z\}e^{i\sqrt{\sigma}t}$

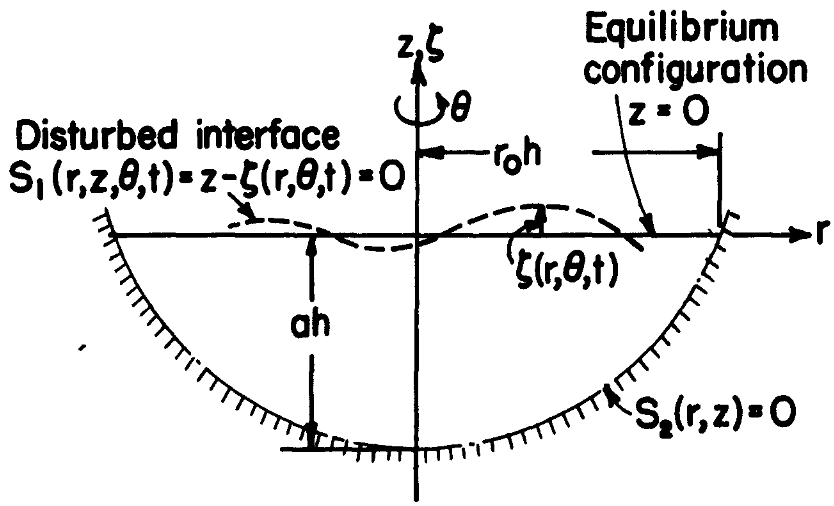


Fig. 5 Reference Frame for Three-Dimensional Layer with Flat Interface.

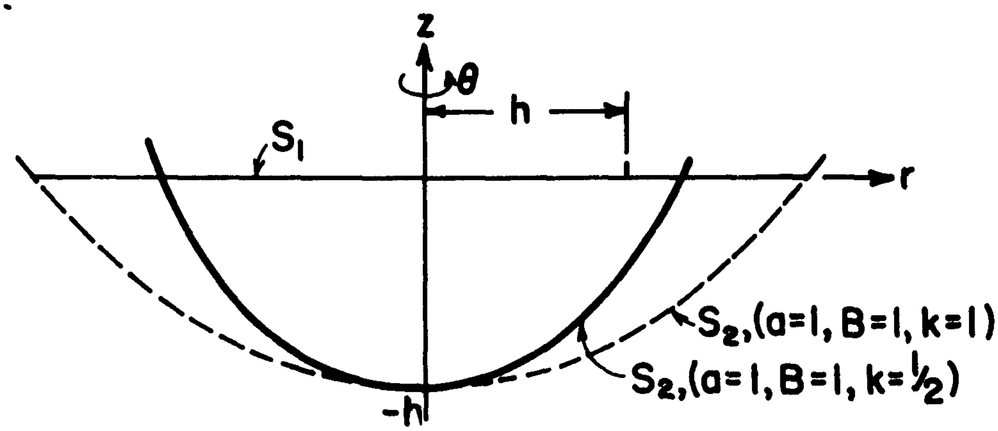


Fig. 6 Three-Dimensional Layers with Free Oscillations Defined by
$\Phi(r,z,\theta,t) = A[\cosh(kz) + B\sinh(kz)] J_1(kr)e^{i(\theta+\sqrt{\sigma}t)}$

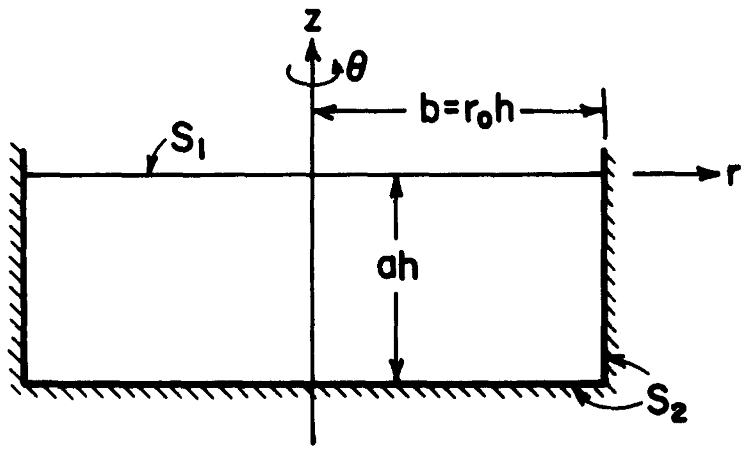


Fig. 7 Three-Dimensional Layer whose Free Oscillations can be Characterized by
$\Phi(r,z,\theta,t) = A\cosh(ka+kz)\, J_m(kr)e^{i(m\theta+\sqrt{\sigma}t)}$
where $J_m(kr_o) = 0$

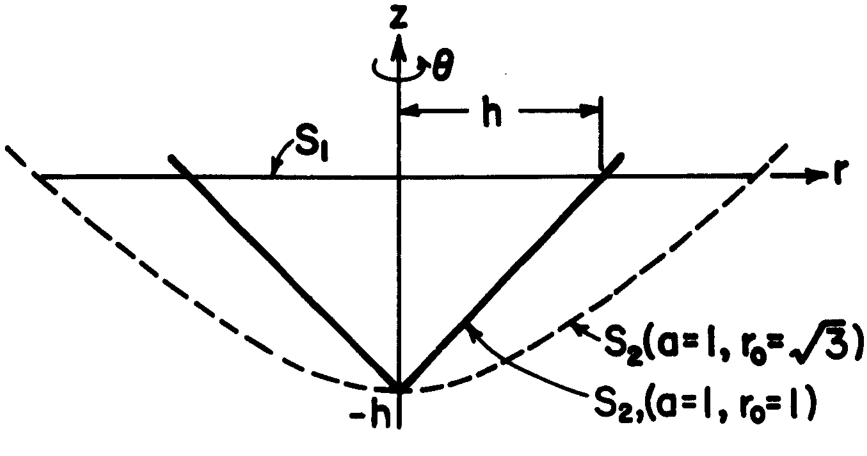


Fig. 8 Three-Dimensional Layers with Free Oscillations Defined by
$\Phi(r,z,\theta,t) = Ar[z+(a^2+r_0^2)/2a]e^{i(\theta+\sqrt{\sigma}t)}$

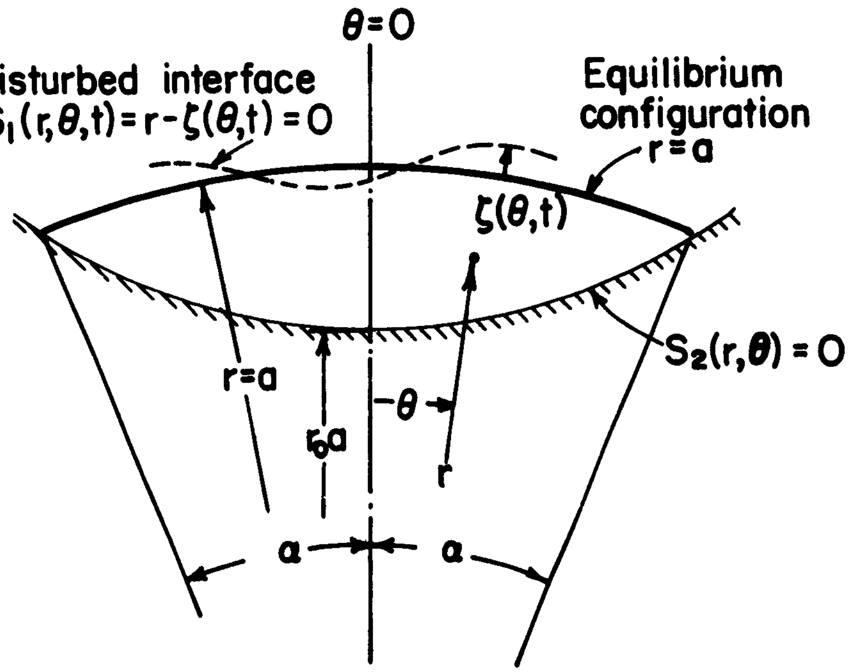


Fig. 9. Reference Frame for Two-Dimensional Layer with Curved Interface.

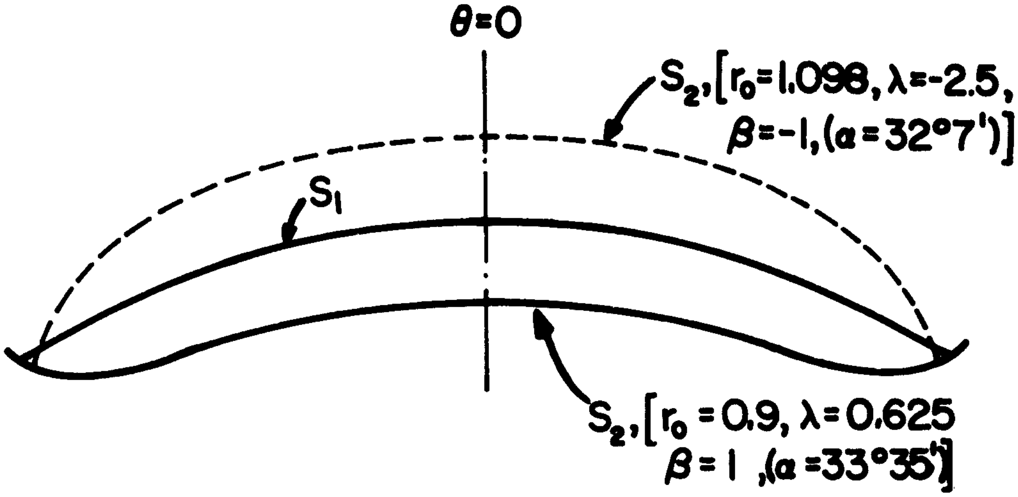
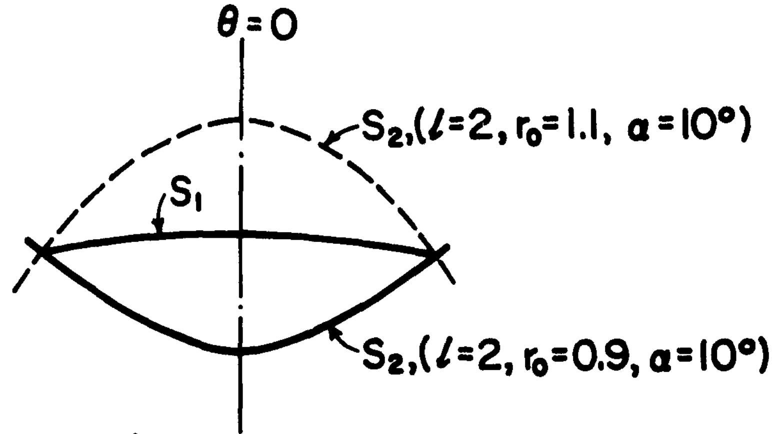
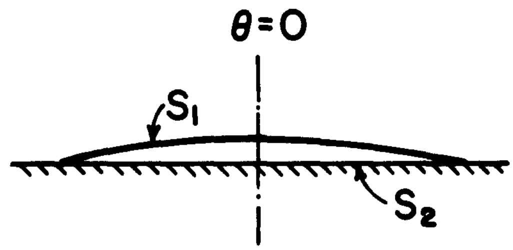


Fig. 10 Two-Dimensional Layers with Oscillations Defined by

$$\Phi(r,\theta,t) = B\left\{\lambda\left(r+\frac{1}{r}\right)\sin\theta + \left[r^2+(1-2\beta\lambda)r^{-2}\right]\sin 2\theta + \lambda(r^3+r^{-3})\sin 3\theta\right\}e^{i\sqrt{\sigma}t}$$ where $\lambda = \left\{1/\left[(\sigma a/g + 6(T/\rho g a^2)\right]\right\}$



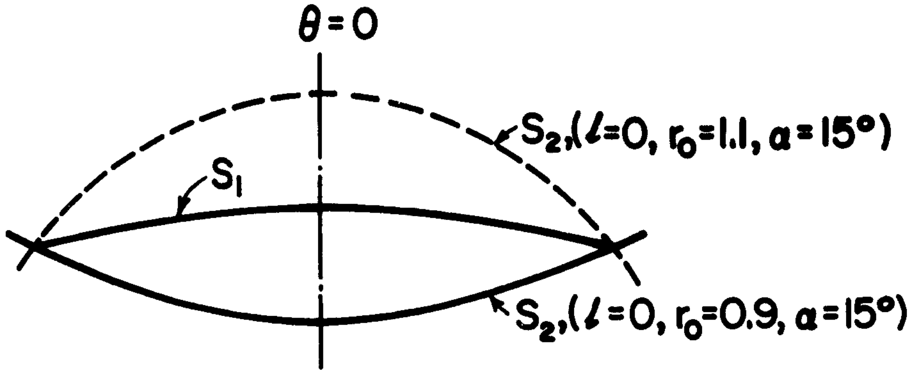
(a) $\Phi = B\left[\left(\frac{r}{r_0}\right)^{\ell} + \left(\frac{r}{r_0}\right)^{-\ell} - \cos \ell\alpha \left(r^{\ell} + r^{-\ell}\right)\right] \sin \ell\theta \, e^{i\sqrt{\sigma}\,t}$



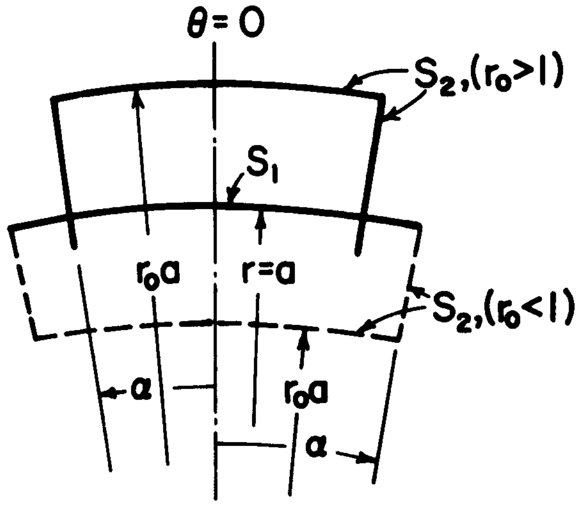
(b) $\ell = 1 \quad r_0 = \cos\alpha$

$\Phi = Br\sin\theta \, e^{i\sqrt{\sigma}\,t}$

Fig. 11a,b. Two-Dimensional Layers with Slightly Curved Interface.



(c) $\ell=0$ $\quad \Phi = B\theta\left[\ell n r - \frac{\alpha^2+(\ell n r_0)^2}{2\ell n r_0}\right]e^{i\sqrt{\sigma}t}$



(d) $\Phi = B\left[\left(\frac{r}{r_0}\right)^{\ell}+\left(\frac{r}{r_0}\right)^{-\ell}\right]\cos\ell\theta\, e^{i\sqrt{\sigma}t}$ where $\ell = \frac{n\pi}{\alpha}$ or

$\Phi = B\left[\left(\frac{r}{r_0}\right)+\left(\frac{r}{r_0}\right)^{-\ell}\right]\sin\ell\theta\, e^{i\sqrt{\sigma}t}$ where $\ell = \frac{(2n+1)\pi}{2\alpha}$

Fig. 11c,d. Two-Dimensional Layers with Slightly Curved Interface

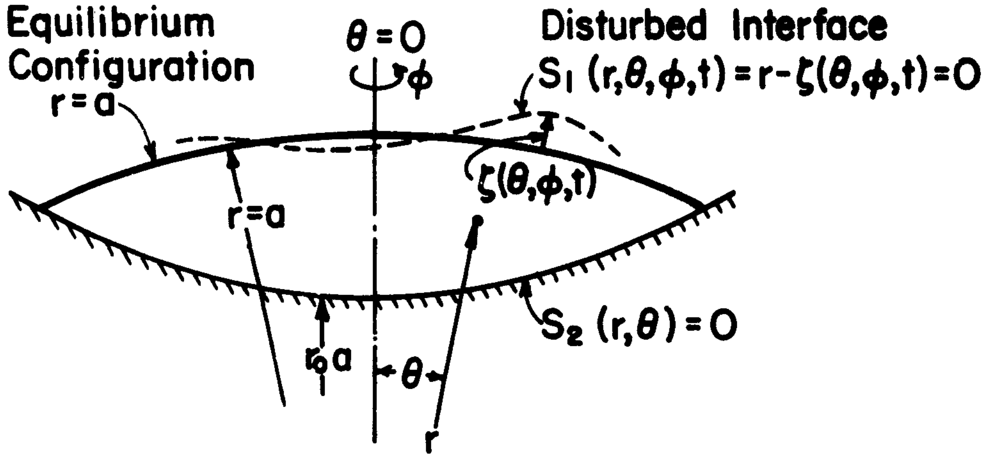


Fig. 12. Reference Frame for Three-Dimensional Layers with Curved Interface.

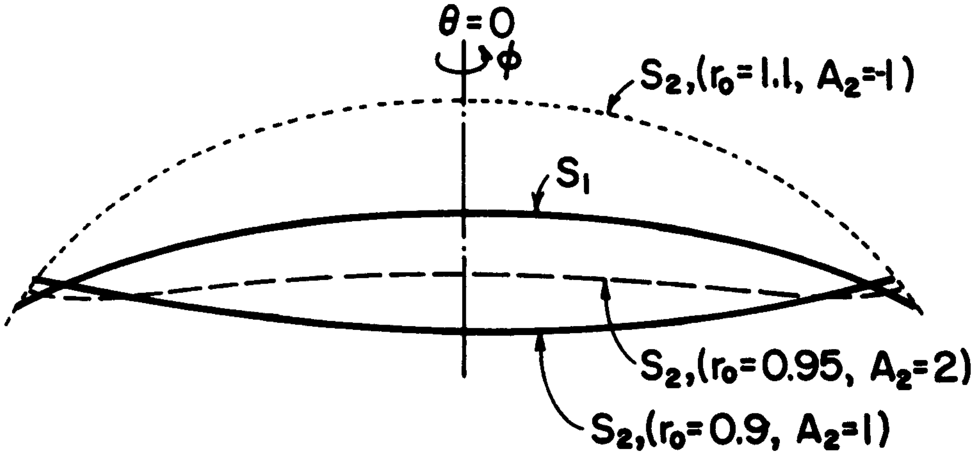


Fig. 13. Three-Dimensional Layers with Oscillations Defined by

$$\Phi = A\left[(r-r^{-2})\sin\theta + A_2(r^2+\tfrac{2}{3}r^{-3})\sin\theta\cos\theta\right]e^{i(\phi+\sqrt{\sigma}t)}.$$

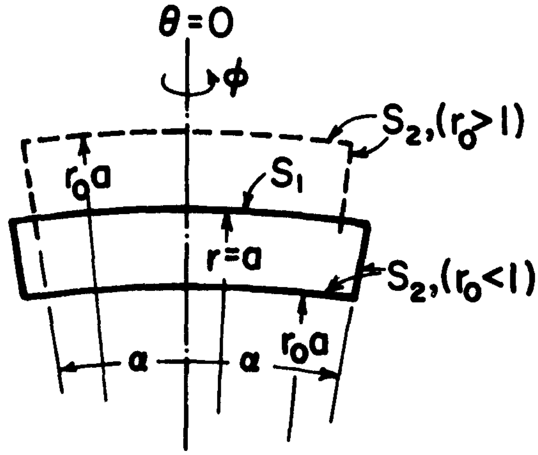
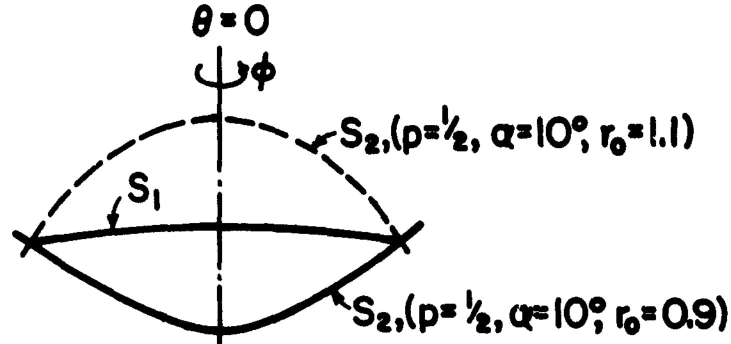


Fig. 14 Three-Dimensional Layers with Slightly Curved Interface and Uniform Thickness. The Fundamental Oscillation of such Layers is Characterized by

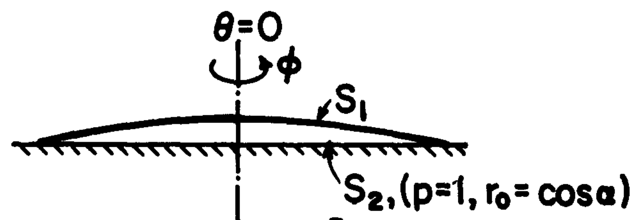
$$\Phi = B\left[r^{p} + \frac{p}{p+1} r_0^{2p+1} r^{-(p+1)}\right] J_m\left[\sqrt{p(p+1)}\,\theta\right] e^{i(m\phi + \sqrt{\sigma} t)}$$

where $J_m'(\sqrt{p(p+1)}\,\alpha) = 0$

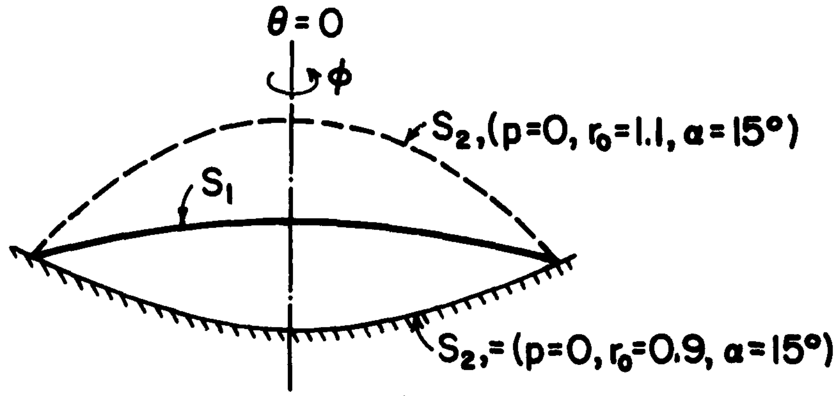


(a) $\Phi = B\left[r^p + A r^{-(p+1)}\right] J_1\left(\sqrt{p(p+1)}\,\theta\right) e^{i(\phi+\sqrt{\sigma}\, t)}$ where

$$A \propto \frac{p\left\{r_0^{p+1} - \left[1-\frac{p(p+1)}{4}\alpha^2\right]^2\right\}}{(p+1)\left\{r_0^{-p} - \left[1-\frac{p(p+1)}{4}\alpha^2\right]^2\right\}} \quad \text{and} \quad \sqrt{p(p+1)}\,\theta \ll 1$$



(b) $\Phi = B r J_1(\sqrt{2}\,\theta)\, e^{i(\phi+\sqrt{\sigma}\, t)}$



(c) $p=0 \quad \Phi = B\theta\left[\frac{1}{r} + \frac{\ln r_0 - \frac{\alpha^2}{2}}{1-r_0}\right] e^{i(\phi+\sqrt{\sigma}\, t)}$

Fig. 15. Three-Dimensional Layers with Slightly Curved Interface and Oscillations Defined as above.

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